NORTH-HOLLAND

Brunowsky Local Form of a Holomorphic Family of Pairs of Matrices

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ABSTRACT

Following Arnold's techniques, we obtain a local canonical form of a holomorphic family of pairs of matrices $(A(\lambda), B(\lambda))$ acted on by the state feedback group. We obtain an explicit formula to compute the dimension of the base space of any miniversal deformation of $(A(0), B(0))$. We make some applications to local perturbations of a pair of matrices. © *Elsevier Science Inc., 1997*

INTRODUCTION

We consider pairs of matrices (A, B) corresponding to a time-invariant linear system $\dot{x} = Ax + Bu$. For convenience, we identify the pair (A, B) with the rectangular block matrix $(A \;\; B)$. We consider the following action of the state feedback group:

$$\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} \cdot (A, B) = P^{-1}(A, B)\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}.$$

LINEAR ALGEBRA AND ITS APPLICATIONS 253:175–198 (1997)

0024-3795/97/$17.00
SSDI 0024-3795(95)00742-3

As in the case of the reduction of a matrix to its Jordan canonical form, the reduction of a pair of matrices to its Brunovsky-Kronecker canonical form is an unstable operation. Following Arnold's technique, the starting point for studying local perturbations, bifurcations diagrams, etc., is the obtaining of a holomorphic canonical form of a local family of pairs, that is to say, the construction of a versal deformation of the central element of the family.

Following this pattern, here we obtain an explicit versal deformation of a pair (A, B) relative to the action above. In fact it is a miniversal deformation with the simplest possible form, in the sense that the number of the nonzero entries is as small as possible.

As an application, we derive a simple proof of the result of J. C. Willems [11, Theorem (6.5)] about the structural stability of a pair (A, B). We also present an interesting example of local perturbation of a particular pair.

Following the same technique of Arnold, Tannenbaum [10, Chapter V] constructed a versal deformation of a pair, but relative to the action of the general linear group via change of basis in the state space.

Notice that the construction of the local transformation of a family into its canonical form lies outside Arnold's technique. For the particular case of families having constant Brunovsky-Kronecker type, see for example [3] or [4].

The organization of this paper is as follows:

In Section I we recall the definition and fundamental properties of the local theory of deformations, and we focus on the situation considered above.

In Section II we construct the above-mentioned miniversal simplest deformation of a pair (II.2.2). In fact, we derive it from an orthogonal miniversal deformation (II.1.5), which presents many more nonzero entries. In particular we give an explicit formula (II.1.9) to compute the dimension of the space of parameters of any miniversal deformation.

Section III contains two applications. In Section III.3 we apply (II.2.2) to prove the structural stability theorem mentioned above, and in Section III.4 we show that the Brunovsky-Kronecker forms which arise from the small perturbations of a pair of the type

$$\left(\left(\begin{array}{c|cc} N & & \\ \hline & \lambda & 1 \\ & & \lambda \end{array}\right), \left(\begin{array}{c} E \\ \hline 0 \\ 0 \end{array}\right)\right)$$

are in correspondence with the Whitney stratification of the classical Whitney umbrella.

Throughout the paper $M_{q\times h}(\mathbf{C})$ denotes the vector space of $q \times h$ matrices with complex entries. If $q = h$ we write $M_q(\mathbf{C})$.

I. PRELIMINARIES

I.1. *The Action of the State Feedback Group*

(I.1.1). We recall that the *state feedback group* is the subgroup of the linear group $\mathrm{Gl}(n+m;\mathbf{C})$ consisting of the matrices of the form

$$\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}$$

where $P \in \mathrm{Gl}(n, \mathbf{C})$, $Q \in \mathrm{Gl}(m, \mathbf{C})$, and $R \in M_{m\times n}(\mathbf{C})$. We will denote this group by $\mathscr{G}$ and its unit element by I. We can identify $\mathscr{G}$ with the open subset $\{(P, Q, R): \det P \neq 0,\ \det Q \neq 0\}$ of the space of triples $M_n(\mathbf{C}) \times M_m(\mathbf{C}) \times M_{m\times n}(\mathbf{C})$, so that $\mathscr{G}$ is a complex manifold.

(I.1.2). We consider the action of $\mathscr{G}$ on $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$

$$\alpha : \mathscr{G} \times \left[M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})\right] \to M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$$

defined by

$$\alpha\left(\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}, (A, B)\right) = P^{-1}(A, B)\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}.$$

The following notation is also used:

$$\alpha\left(\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}, (A, B)\right) = \begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} \cdot (A, B).$$

If we fix the pair (A, B), then $\alpha_{(A,B)}$ is the map

$$\alpha_{(A,B)} : \mathscr{G} \to M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$$

such that

$$\alpha_{(A,B)}\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} = \alpha\left(\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}, (A, B)\right).$$

It is clear that the mappings α and $\alpha_{(A,B)}$ are holomorphic. Moreover, $\alpha_{(A,B)}$ is a rational map.

(I.1.3). The action defined by α induces the following equivalence relation between pairs of matrices: (A, B) and (C, D) are called *block-similar* if and only if there exists $g \in \mathscr{G}$ such that $\alpha_{(A,B)}(g) = (C, D)$.

Then the equivalence class of (A, B) is its orbit under the action of α, $\alpha_{(A,B)}(\mathscr{G})$, which we shall denote by $\mathscr{O}(A, B)$.

(I.1.4)

PROPOSITION. *The orbits $\mathscr{O}(A, B)$ are complex submanifolds of $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$.*

Proof. We follow the reasoning of [7]. Since $\alpha_{(A,B)}$ is a rational map and $\mathscr{G}$ is obviously a constructible set, Chevalley's theorem (see for example [8, Theorem (4.4)] states that $\alpha_{(A,B)}(\mathscr{G}) = \mathscr{O}(A, B)$ is also constructible. Then $\mathscr{O}(A, B)$ has a nonsingular point. Taking into account that given any two points on the orbit there is a diffeomorphism of $M_n(\mathbf{C}) \times M_{m\times n}(\mathbf{C})$ preserving the orbit and mapping one onto the other, it follows that every point on the orbit is nonsingular. Hence $\mathscr{O}(A, B)$ is a complex manifold. ∎

(I.1.5). We denote by $\mathscr{St}(A, B)$ the *stabilizer* of (A, B) under the action of $\mathscr{G}$; that is to say, the closed subgroup of $\mathscr{G}$ defined by

$$\mathscr{St}(A, B) = \left\{ \begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} \in \mathscr{G};\ \alpha_{(A,B)}\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} = (A, B) \right\}.$$

It is well known that $\mathscr{St}(A, B)$ is a complex manifold. In our case it is obvious, since $\mathscr{St}(A, B)$ is the intersection of the open set $\mathscr{G}$ with a linear variety.

Also, the homogeneous manifold $\mathscr{G}/\mathscr{St}(A, B)$ is diffeomorphic to the manifold $\mathscr{O}(A, B)$, so that

$$\begin{aligned} \dim \mathscr{O}(A, B) &= \dim \mathscr{G} - \dim \mathscr{St}(A, B) \\ &= n^2 + m^2 + nm - \dim \mathscr{St}(A, B) \end{aligned}$$

(I.1.6). If $d\alpha_{(A,B),I}$ is the differential mapping of $\alpha_{(A,B)}$ at the point I, one has:

(i) $\operatorname{Ker} d\alpha_{(A,B),I} \cap \mathscr{G} = \mathscr{St}(A, B)$,
(ii) $\operatorname{Im} d\alpha_{(A,B),I} = T_{(A,B)}\mathscr{O}(A, B)$.

(I.1.7). We shall use an explicit description of $d\alpha_{(A,B),I}$. It can be derived from (i) above. We also present a direct computation.

Recall that, since $\mathscr{G}$ is an open subset of $M_n(\mathbf{C}) \times M_m(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$, the tangent space $T_I\mathscr{G}$ is $M_n(\mathbf{C}) \times M_m(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$.

LEMMA. *With the notation above,*

$$d\alpha_{(A,B),I}\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} = ([A,P] + BR, BQ - PB).$$

Proof. We compute

$$\alpha_{(A,B)}\left(I + \varepsilon\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}\right)$$

where $\varepsilon \in \mathbf{R}$ is small enough:

$$\begin{aligned}
&\alpha_{(A,B)}\left(I + \varepsilon\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}\right)\\
&= \alpha_{(A,B)}\begin{pmatrix} I + \varepsilon P & 0 \\ \varepsilon R & I + \varepsilon Q \end{pmatrix}\\
&= (I + \varepsilon P)^{-1}(A, B)\begin{pmatrix} I + \varepsilon P & 0 \\ \varepsilon R & I + \varepsilon Q \end{pmatrix}\\
&= (I - \varepsilon P + \varepsilon^2 P^2 - \cdots)(A, B)\begin{pmatrix} I + \varepsilon P & 0 \\ \varepsilon R & I + \varepsilon Q \end{pmatrix}\\
&= (A, B) + \varepsilon([A,P] + BR, BQ - PB) + \varepsilon^2(\cdots) + \cdots .
\end{aligned}$$

Then the lemma follows easily. ∎

I.2. *Versal Deformations*

(I.2.1). Let $(A, B) \in M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$, and Λ be a neighborhood of the origin in $\mathbf{C}^l$. A *deformation* of (A, B) is a holomorphic mapping

$$\varphi : \Lambda \to M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$$

such that $\varphi(0) = (A, B)$. We set $\varphi(\lambda) = (A(\lambda), B(\lambda))$ for $\lambda \in \Lambda$; then we call $\{(A(\lambda), B(\lambda))\}_{\lambda\in\Lambda}$ a *family of deformations* of (A, B).

Usually, the set Λ is called the *base* of the deformation φ, and if $\lambda = (\lambda_1, \dots, \lambda_l) \in \Lambda$, we say that each λ_l is a *parameter* of the deformation.

(I.2.2). Let Γ be a neighborhood of the origin in $\mathbf{C}^k$, φ a deformation of (A, B), and $\theta : \Gamma \to \Lambda$ a holomorphic map such that $\theta(0) = 0$. Then the *deformation induced* by θ is the map

$$\theta^*\varphi : \Gamma \to M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$$

defined by $\theta^*\varphi = \varphi \circ \theta$, that is to say,

$$(\theta^*\varphi)(\mu) = \varphi(\theta(\mu)), \qquad \mu \in \Gamma,$$

and $\{(A(\theta(\mu)), B(\theta(\mu)))\}_{\mu\in\Gamma}$ is the *family of deformations* of (A, B) *induced* by θ.

(I.2.3). A deformation of (A, B), $\varphi : \Lambda \to M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$, is called *versal* at 0 if for any deformation $\psi : \Gamma \to M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ of (A, B) there exists an open set $\Gamma' \subset \Gamma$ with $0 \in \Gamma'$, a holomorphic map $\theta : \Gamma \to \Lambda$ with $\theta(0) = 0$, and a deformation $\gamma : \Gamma' \to \mathcal{G}$ of $I \in \mathcal{G}$ such that

$$\psi(\mu) = \gamma(\mu) \cdot \varphi(\theta(\mu)), \qquad \mu \in \Gamma',$$

that is to say, if

$$(A(\mu), B(\mu)) = P^{-1}(\mu)(A(\theta(\mu)), B(\theta(\mu)))\begin{pmatrix} P(\mu) & 0 \\ R(\mu) & Q(\mu) \end{pmatrix},$$

$$\mu \in \Gamma'$$

(I.2.4). As Tannenbaum pointed out [10, Theorem (V.1.2)], the geometric characterization of the versality condition given by Arnold [1] holds in the general situation where $\mathcal{G}$ is a complex Lie group acting on a complex manifold.

For the convenience of the reader we will outline the proof in our particular setting. It is based on the following:

(I.2.5)

LEMMA. *Let* $\varphi : \Lambda \to M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ *be a deformation of* (A, B) *transversal to the orbit* $\mathcal{O}(A, B)$ *at* 0, *with* $\dim \Lambda = n^2 + nm -$

dim $\mathscr{O}(A, B)$, *and* $V \subset \mathscr{G}$ *a submanifold which is transversal to* $\mathscr{S}t(A, B)$ *at* I, *with* $\dim V = \dim \mathscr{O}(A, B)$. *Then the mapping*

$$\beta : \Lambda \times V \to M_n(\mathbf{C}) \times M_{n \times m}(\mathbf{C})$$

defined by

$$\beta\left(\lambda, \begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}\right) = P^{-1}(A(\lambda), B(\lambda))\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}$$

is a local diffeomorphism at $(0, I)$.

We recall (see for example [9]) that the condition of transversality means that

$$\operatorname{Im} d\varphi_0 + T_{(A, B)}\mathscr{O}(A, B) = M_n(\mathbf{C}) \times M_{n \times m}(\mathbf{C}). \qquad (*)$$

Proof. Since $\dim(V \times \Lambda) = n^2 + nm$, it is enough to see that $d\beta_{(0, I)}$ is surjective, and this follows from the conditions imposed on φ, Λ, and V, and (I.1.6). ∎

(I.2.6)

PROPOSITION. *A deformation* $\varphi : \Lambda \to M_n(\mathbf{C}) \times M_{n \times m}(\mathbf{C})$ *of* (A, B) *is versal at* 0 *if and only if it is transversal to the orbit* $\mathscr{O}(A, B)$.

Proof. Assume that φ is versal at 0. We have to show the equality $(*)$. Take $(C, D) \in M_n(\mathbf{C}) \times M_{n \times m}(\mathbf{C})$, and consider the deformation of (A, B)

$$\psi : \mathbf{C} \to M_n(\mathbf{C}) \times M_{n \times m}(\mathbf{C})$$

defined by $\psi(\mu) = (A, B) + \mu(C, D)$.

Since φ is versal, then

$$\psi(\mu) = P^{-1}(\mu)(A(\theta(\mu)), B(\theta(\mu)))\begin{pmatrix} P(\mu) & 0 \\ R(\mu) & Q(\mu) \end{pmatrix},$$

and computing $d\psi_0$, it follows that

$$(C, D) \in \operatorname{Im} d\alpha_{(A,B),I} + \operatorname{Im} d\varphi_0 = T_{(A,B)}\mathscr{O}(A, B) + \operatorname{Im} d\varphi_0.$$

Conversely, let φ be a deformation of (A, B) as in the lemma. Then, if

$$\psi : \Gamma \to M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$$

is a deformation of (A, B), and μ is small enough, we have

$$\psi(\mu) = \beta\left(\theta(\mu), \begin{pmatrix} P(\mu) & 0 \\ R(\mu) & Q(\mu) \end{pmatrix}\right),$$

where

$$\theta(\mu) = \pi_1 \beta^{-1}\psi(\mu) \quad \text{and} \quad \begin{pmatrix} P(\mu) & 0 \\ R(\mu) & Q(\mu) \end{pmatrix} = \pi_2 \beta^{-1}\psi(\mu),$$

π_1 and π_2 being the natural projections on V and Λ, respectively. ■

II. CONSTRUCTION OF MINIVERSAL DEFORMATIONS

II.1. First Miniversal Deformation

Firstly, we construct a miniversal deformation as a submanifold of $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ orthogonal to $\mathscr{O}(A, B)$ with respect to an appropriate hermitian product.

(II.1.1). A versal deformation of (A, B) is called *miniversal* if it has the minimum number of parameters among all the versal deformations of (A, B).

Assume that we have a hermitian scalar product in $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$, and let $(U^i, V^i)_{1\leq i\leq d}$ be a basis of $[T_{(A,B)}\mathscr{O}(A, B)]^{\perp}$. Then according to (I.2.6) the mapping

$$\varphi : \mathbf{C}^d \to M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$$

defined by

$$\varphi(\lambda_1, \ldots, \lambda_d) = (A, B) + \sum_{i=1}^{d} \lambda_i (U^i, V^i)$$

is a miniversal deformation of (A, B) at 0.

(II.1.2). We shall consider in $M_n(\mathbf{C}) \times M_{n\times m}(\mathbf{C})$ the hermitian scalar product defined by

$$\langle (X,Y),(U,V)\rangle = \operatorname{tr}[(X,Y)\cdot(U,V)^*],$$

where

$$(U,V)^* = \begin{pmatrix} \overline{U}^t \\ \overline{V}^t \end{pmatrix}$$

is the adjoint of (U,V) and "tr" stands for trace.

(II.1.3). The following lemma gives a useful condition for obtaining $[T_{(A,B)}\mathscr{O}(A,B)]^{\perp}$.

LEMMA. *With the above notation,* $(U,V) \in [T_{(A,B)}\mathscr{O}(A,B)]^{\perp}$ *if and only if*

$$\left.\begin{aligned} [A,U^*] + BV^* &= 0 \\ U^*B &= 0 \\ V^*B &= 0 \end{aligned}\right\}.$$

Proof. $(U,V) \in [T_{(A,B)}\mathscr{O}(A,B)]^{\perp}$ if and only if for any

$$\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix} \in T_I\mathscr{G}$$

one has

$$\left\langle d\alpha_{(A,B),I}\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix},(U,V)\right\rangle = 0.$$

From (I.1.7) it follows that this condition is equivalent to

$$\langle([A,P] + BR, BQ - PB),(U,V)\rangle = 0.$$

It can be checked that

$$\begin{aligned} &\operatorname{tr}\{([A,P] + BR, BQ - PB),(U,V)(U,V)^*\} \\ &\quad = \operatorname{tr}\left[\begin{pmatrix} [A,U^*] + BV^* & -U^*B \\ 0 & -V^*B \end{pmatrix}\begin{pmatrix} P & 0 \\ R & Q \end{pmatrix}\right] \\ &\quad = \operatorname{tr}\left[\begin{pmatrix} [A,U^*] + BV^* & -U^*B \\ 0 & -V^*B \end{pmatrix}\begin{pmatrix} P & S \\ R & Q \end{pmatrix}\right], \end{aligned}$$

because S does not play any role for computing the trace. So $(U,V) \in [T_{(A,B)}\mathscr{O}(A,B)]^{\perp}$ if and only if

$$\operatorname{tr}\left[\begin{pmatrix} [A,U^*]+BV^* & -U^*B \\ 0 & -V^*B \end{pmatrix}\begin{pmatrix} P & S \\ R & Q \end{pmatrix}\right] = 0.$$

Since

$$\begin{pmatrix} P & S \\ R & Q \end{pmatrix} \in M_{n+m}(\mathbf{C})$$

is arbitrary, this condition is equivalent to the ones stated in the lemma. ■

II.1.4. If (A,B) is in the Brunovsky-Kronecker form, it is possible to write down explicitly the miniversal deformation of (A,B) considered in (I.1.1) and in particular to compute its number of parameters. In order to do this, we recall that a Brunovsky-Kronecker matrix has the form (see [5, Theorem (6.2.5)])

$$A = \begin{pmatrix} N & 0 \\ 0 & J \end{pmatrix}, \qquad B = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix},$$

where

(a) $N = \operatorname{diag}(N_1, \dots, N_r)$ with

$$N_i = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in M_{k_i}(\mathbf{C}), \qquad 1 \leqslant i \leqslant r$$

(we assume that $k_1 \geqslant \cdots \geqslant k_r$, and we write $p = k_1 + \cdots + k_r$),

(b) $J = \operatorname{diag}(J_1, \dots, J_s)$ with

$$J_i = \begin{pmatrix} \lambda_j & 1 & 0 & \cdots & 0 \\ 0 & \lambda_j & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_j \end{pmatrix} \in M_{l_j}(\mathbf{C}), \qquad 1 \leqslant j \leqslant s,$$

and

(c) $E = \operatorname{diag}(E_1, \dots, E_r)$ with $E_i = (0, \dots, 0, 1)^t \in M_{k_i \times 1}(\mathbf{C})$, $1 \leqslant i \leqslant r$.

(II.1.5). Now, we partition the matrices U^* and V^* in (II.1.3) into blocks

$$U^* = \begin{pmatrix} U_1^1 & U_2^1 \\ U_1^2 & U_2^2 \end{pmatrix}, \qquad V^* = \begin{pmatrix} V_1^1 & V_2^1 \\ V_1^2 & V_2^2 \end{pmatrix}$$

according to the partition of the Brunovsky-Kronecker matrix (A, B), so that the systems of the lemma (II.1.3) split into the following systems:

$$\left.\begin{aligned} [N, U_1^1] + EV_1^1 &= 0 \\ U_1^1 E &= 0 \\ V_1^1 E &= 0 \end{aligned}\right\}, \tag{I}$$

$$\left.\begin{aligned} JU_1^2 - U_1^2 N &= 0 \\ U_1^2 E &= 0 \end{aligned}\right\}, \tag{II}$$

$$NU_2^1 - U_2^1 J + EV_2^1 = 0, \tag{III}$$

$$V_1^2 E = 0, \tag{IV}$$

$$[J, U_2^2] = 0, \tag{V}$$

the matrix V_2^2 being arbitrary.

In order to solve these systems we again decompose the unknown matrices $U_1^1, U_2^1, \ldots, V_1^1, \ldots$ into blocks, this time according to the block decomposition of N, J, and E. Then, denoting now by U and V the (i, j) block of any of the above unknown matrices, we reduce this to solving the following systems:

$$\left.\begin{aligned} N_i - UN_j + E_i V &= 0 \\ UE_j &= 0 \\ VE_j &= 0 \end{aligned}\right\}, \tag{1}$$

$$\left.\begin{aligned} J_i U - UN_j &= 0 \\ UE_j &= 0 \end{aligned}\right\}, \tag{2}$$

$$N_i U - UJ_j + E_i V = 0, \tag{3}$$

$$VE_j = 0, \tag{4}$$

$$[J, U] = 0. \tag{5}$$

(II.1.6). The system (5) is solved in [6, Chapter VIII]. By means of similar techniques we obtain the following explicit description of the solutions of the other systems:

PROPOSITION. *Let $\mathcal{S}_1, \dots, \mathcal{S}_4$ be the spaces of solutions of the system* (1), ..., (4), *respectively. Then:*

(i) $(U, V) \in \mathcal{S}_1$ *if and only if*

(α) *if* $k_j \leqslant k_i + 1$, *then* $U = 0$, $V = 0$;

(β) *if* $k_j \geqslant k_i + 2$, *then*

$$U = \begin{pmatrix} v_1 & v_2 & \cdots & \cdots & v_{k_j - k_i - 1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & v_1 & \cdots & \cdots & v_{k_j - k_i - 2} & v_{k_j - k_i - 1} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & v_1 & \cdots & \cdots & \cdots & v_{k_j - k_i - 1} & 0 & 0 \end{pmatrix}$$

$$\in M_{k_i \times k_j}(\mathbf{C}),$$

$$V = \begin{array}{c} \begin{array}{cccccc} & & & k_i+1 & & \end{array} \\ \begin{pmatrix} 0 & \cdots & 0 & v_i & \cdots & v_{k_j - k_i - 1} & 0 \end{pmatrix} \end{array} \in M_{1 \times k_j}(\mathbf{C}).$$

(ii) $U \in \mathcal{S}_2$ *if and only if* $U = 0$.

(iii) $(U, V) \in \mathcal{S}_3$ *if and only if the coefficients of* U *and* V *can be obtained from* $u_1^1, \dots, u_l^1$ *by means of the following set of recurrent relations*:

$$\begin{Bmatrix} u_1^2 = \lambda u_1^1 \\ u_2^2 = u_1^1 + \lambda u_2^1 \\ \vdots \\ u_l^2 = u_{l-1}^1 + \lambda u_l^1 \end{Bmatrix}, \dots, \begin{Bmatrix} u_1^k = \lambda u_1^{k-1} \\ u_2^k = u_1^{k-1} + \lambda u_2^{k-1} \\ \vdots \\ u_k^l = u_{l-1}^{k-1} + \lambda u_l^{k-1} \end{Bmatrix}, \quad \begin{Bmatrix} v_1 = \lambda u_1^k \\ v_2 = u_1^k + \lambda u_2^k \\ \vdots \\ v_l = u_{l-1}^k + \lambda u_l^k \end{Bmatrix},$$

where $U = (u_s^r)$, $V = (v_s)$, $1 \leqslant r \leqslant k$, $1 \leqslant s \leqslant l$. *(We have written* k *and* l *instead of* k_i *and* l_j. *Observe that the parameters* $u_1^1, \dots, u_l^1$ *are independent.)*

(iv) $V \in \mathcal{S}_4$ *if and only*

$$V = \begin{pmatrix} v_1 & \cdots & v_{k_j - 1} & 0 \end{pmatrix}.$$

(II.1.7). Counting the independent coefficients in the unknown matrices, we obtain:

COROLLARY. *With the notation of the preceding proposition, we have that*

(i) $\dim \mathscr{S}_1 = \max\{0, k_j - k_i - 1\}$,
(ii) $\dim \mathscr{S}_2 = 0$,
(iii) $\dim \mathscr{S}_3 = l_j$,
(iv) $\dim \mathscr{S}_4 = k_j - 1$.

(II.1.8). From (II.1.7) and [1] we have

PROPOSITION. *Let* $\mathscr{S}_{\mathrm{I}}, \dots, \mathscr{S}_{\mathrm{V}}$ *be the spaces of solutions of the systems* (I), ..., (V) *in* (II.1.5). *Then*

(i) $\dim \mathscr{S}_{\mathrm{I}} = \sum_{1 \leqslant i, j \leqslant r} \max\{0, k_j - k_i - 1\}$,
(ii) $\dim \mathscr{S}_{\mathrm{II}} = 0$,
(iii) $\dim \mathscr{S}_{\mathrm{III}} = r(n - p)$,
(iv) $\dim \mathscr{S}_{\mathrm{IV}} = (m - r)(p - r)$,
(v) $\dim \mathscr{S}_{\mathrm{V}} = \sum_\lambda (\delta_1(\lambda) + 3\delta_2(\lambda) + 5\delta_3(\lambda) + \cdots)$,

where $\delta_1(\lambda) \geqslant \delta_2(\lambda) \geqslant \delta_3(\lambda) \geqslant \cdots$ *is the Segrè characteristic of the eigenvalue* λ, *which runs over the set of all the eigenvalues of* J.

(II.1.9). From the above proposition we obtain immediately

PROPOSITION. *The dimension of the base space of any miniversal deformation of* (A, B) *at* 0 *is given by*

$$\dim \left[T_{(A,B)}\mathscr{O}(A, B)\right]^{\perp}$$

$$= \sum_{1 \leqslant i, j \leqslant r} \max\{0, k_j - k_i - 1\} + r(n - p) + (m - r)(p - r)$$

$$+ \sum_{\lambda} (\delta_1(\lambda) + 3\delta_2(\lambda) + 5\delta_3(\lambda) + \cdots) + (m - r)(n - p).$$

(II.1.10)

EXAMPLE. Let (A, B) be the pair

$$A = \left(\begin{array}{ccccccc|cc}
0&1&0&0&&&&&\\
0&0&1&0&&&&&\\
0&0&0&1&&&&&\\
0&0&0&0&&&&&\\
&&&&0&1&&&\\
&&&&0&0&&&\\
&&&&&&0&&\\ \hline
&&&&&&&\lambda&1\\
&&&&&&&&\lambda
\end{array}\right), \qquad B = \left(\begin{array}{ccc|c}
0&&&\\
0&&&\\
0&&&\\
1&&&\\
&0&&\\
&1&&\\
&&1&\\ \hline
&&&\\
&&&
\end{array}\right)$$

with $n = 9$, $m = 4$, $r = 3$, $k_1 = 4$, $k_2 = 2$, $k_3 = 1$, $l_1 = 2$.

From (II.1.6) it follows that $(U, V) \in [T_{(A,B)}\mathscr{O}(A, B)]^{\perp}$ if and only if it is of the form

$$U = \left(\begin{array}{cccc|cc|c|cc}
&&&&v_2^3&0&v_3^2&&\\
&&&&0&v_2^3&v_3^3&&\\
&&&&0&0&0&&\\
&&&&0&0&0&&\\ \hline
&&&&&&&&\\
&&&&&&&&\\ \hline
&&&&&&&&\\ \hline
u_1^8&\lambda u_1^8&\lambda^2 u_1^8&\lambda^3 u_1^8&u_5^8&\lambda u_5^8&u_7^8&u_8^8&0\\
u_1^9&u_1^8+\lambda u_1^9&2\lambda u_1^8+\lambda^2 u_1^9&3\lambda^2 u_1^9+\lambda^3 u_1^9&u_5^9&u_5^8+\lambda u_5^9&u_7^9&u_8^9&u_8^8
\end{array}\right),$$

$$V = \left(\begin{array}{c|c|c|c}
0&0&0&v_4^1\\
0&0&v_3^2&v_4^2\\
0&v_2^3&v_3^3&v_4^3\\
0&0&0&0\\ \hline
&0&0&v_4^5\\
&0&0&0\\ \hline
&&0&0\\ \hline
&&&v_4^8\\
&&&v_4^9
\end{array}\right).$$

Then we obtain a basis of $[T_{(A,B)}\mathscr{O}(A, B)]^{\perp}$ by setting one of the parameters equal to 1, the remaining ones being 0. For instance, if $u_1^8 = 1$ and the other parameters are 0, we obtain

$$(U^1, V^1) = \left(\left(\begin{array}{cccc|c|c|c} &&&&&& \\ \hline &&&&&& \\ &&&&&& \\ \hline &&&&&& \\ &&&&&& \\ \hline 1 & \lambda & \lambda^2 & \lambda^3 &&& \\ 0 & 1 & 2\lambda & 3\lambda^2 &&& \end{array} \right), \begin{pmatrix} \\ 0 \\ \\ \end{pmatrix} \right),$$

Notice that, according to (II.1.9), $\dim [T_{(A,B)}\mathscr{O}(A, B)]^{\perp} = 17$.

(II.1.11). It is clear that the procedure given in the above example for obtaining a basis of $[T_{(A,B)}\mathscr{O}(A, B)]^{\perp}$ is a general one: A basis is obtained in a similar way, setting all the independent parameters to zero except one of them, in the set of matrices (U, V) given by

$$U = \begin{pmatrix} (U_1^1)^* & (U_1^2)^* \\ (U_2^1)^* & (U_2^2)^* \end{pmatrix}, \qquad V_2 = \begin{pmatrix} (V_1^1)^* & (V_1^2)^* \\ (V_2^1)^* & (V_2^2)^* \end{pmatrix}$$

where the (i, j) blocks of $U_1^1, U_1^2, \ldots, V_1^2, \ldots$ are the solutions obtained in (II.1.6).

II.2. Second Miniversal Deformation

Now, we are going to obtain a new miniversal deformation $(A, B) + (X, Y)$ of (A, B) in which the number of nonzero entries of (X, Y) will be minimal, that is, only the number of independent parameters.

(II.2.1). From (I.2.6) it follows that in order to obtain a miniversal deformation of (A, B) it is sufficient to get a linear variety defined by a supplementary subspace to $T_{(A,B)}\mathscr{O}(A, B)$ in $M_n(\mathbf{C}) \times M_{n \times m}(\mathbf{C})$. To this end we characterize $T_{(A,B)}\mathscr{O}(A, B)$ by means of the basis $(U^i, V^i)_{1 \leqslant i \leqslant d}$ of $[T_{(A,B)}\mathscr{O}(A, B)]^{\perp}$ in (II.1.11) as follows: $(C, D) \in M_n(\mathbf{C}) \times M_{n \times m}(\mathbf{C})$ belongs to $T_{(A,B)}\mathscr{O}(A, B)$ if and only if

$$\operatorname{tr}\left[(U^i, V^i) \begin{pmatrix} C^* \\ D^* \end{pmatrix} \right] = 0, \qquad 1 \leqslant i \leqslant d,$$

Therefore, to obtain a basis of a miniversal deformation of (A, B) at 0, it is sufficient to take the linear variety defined by a set of pairs (X, Y) with exactly d parameters $\mu_1, \ldots, \mu_d$, each one placed in only one entry in such a way that

$$\operatorname{tr}\left[(U^i, V^i)\begin{pmatrix} X^* \\ Y^* \end{pmatrix}\right] = \mu_i, \qquad 1 \leqslant i \leqslant d.$$

Then $(A, B) + (X, Y)$ will be a miniversal deformation verifying the above condition.

We are going to illustrate the method of obtaining such a pair (X, Y) using the example (II.1.10).

Firstly, we order the basis (U^i, V^i) in the following natural way: $(U^1, V^1), (U^2, V^2), \ldots, (U^{17}, V^{17})$ are the elements of the basis obtained by making 0 all the parameters of (U, V) except $u_1^8, u_1^9, u_5^8, u_5^9, \ldots, u_8^8, u_8^9, v_2^3, v_3^2, v_3^3, v_4^1, v_4^2, \ldots, v_4^9$, which are respectively equal to 1. Then, if we consider the pair

$$X = \left(\begin{array}{ccccccc|cc} & & & & & & & & \\ & & & & & & & & \\ \hline x_1^8 & 0 & 0 & 0 & x_5^8 & 0 & x_7^8 & 0 & 0 \\ x_1^9 & 0 & 0 & 0 & x_5^9 & 0 & x_7^9 & x_8^9 & x_9^9 \end{array}\right),$$

$$Y = \begin{pmatrix} 0 & 0 & 0 & y_4^1 \\ 0 & 0 & y_3^2 & y_4^2 \\ 0 & y_2^3 & y_3^3 & y_4^3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_4^5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y_4^8 \\ 0 & 0 & 0 & y_4^9 \end{pmatrix},$$

it is easy to check that

$$\langle (A_1^1, B_1^1), (X, Y) \rangle = \bar{x}_1^8,$$

$$\langle (A_1^2, B_1^2), (X, Y) \rangle = \bar{x}_1^9,$$

$$\langle (A_1^3, B_1^3), (X, Y) \rangle = \bar{x}_5^8,$$

$$\vdots$$

$$\langle (A_1^9, B_1^9), (X, Y) \rangle = \bar{y}_2^3,$$

$$\vdots$$

so the above pair has the desired property.

(II.2.2). In general one has the following

THEOREM. *Let (A, B) be a pair in the Brunovsky-Kronecker form, and (X, Y) the linear variety of $M_n(\mathbf{C}) \times M_{n \times m}(\mathbf{C})$ defined by*

$$X = \begin{pmatrix} X_1^1 & X_2^1 \\ X_1^2 & X_2^2 \end{pmatrix}, \qquad Y = \begin{pmatrix} Y_1^1 & Y_2^1 \\ Y_1^2 & Y_2^2 \end{pmatrix},$$

where:

(i) $X_1^1 = 0$, $X_2^1 = 0$.
(ii) *All the entries of X_1^2 are zero except the ones corresponding to*

$$x_1^{p+1}, \ldots, x_1^n, x_{k_1+1}^{p+1}, \ldots, x_{k_1+1}^n, \ldots, x_{k_1+\cdots+k_{r-1}+1}^{p+1}, \ldots, x_{k_1+\cdots+k_{r-1}+1}^n,$$

which are arbitrary.

(iii) $J + X_2^2$ *is the Arnold canonical form* ([1, *Theorem* (4.4)]; *see the remark* (II.2.3) *below*).

(iv) *All the entries of* Y_1^1 *are zero except the ones corresponding to*

$$y_2^{k_2+1},\dots,y_2^{k_1-1},y_3^{k_1+k_3+1},\dots,y_k^{k_1+\cdots+k_{r-2}+k_r+1},\dots,y_r^{k_1+\cdots+k_{r-1}-1},$$

which are arbitrary.

(v) Y_2^1 *is such that*

$$y_{r+1}^{k_1}=\cdots=y_m^{k_1}=y_{r+1}^{k_1+k_2}=\cdots=y_m^{k_1+k_2}=\cdots=y_{r+1}^{p}=\cdots=y_m^{p}=0.$$

(vi) $Y_1^2=0$.

(vii) *All the parameters in* Y_2^2 *are arbitrary.*

Then $(A,B)+(X,Y)$ *is a miniversal deformation of* (A,B) *at* 0 *such that the nonzero entries of* (X,Y) *are just the independent parameters.*

(II.2.3)

REMARK. We recall that if, for example, J is a Jordan block of the form

$$J=\begin{pmatrix}\lambda & 1 & & & & & \\ & \lambda & 1 & & & & \\ & & \lambda & 1 & & & \\ & & & \lambda & & & \\ & & & & \lambda & 1 & \\ & & & & & \lambda & \\ & & & & & & \lambda\end{pmatrix},$$

then X_2^2 in (iii) of (II.2.2) is

$$X_2^2=\begin{pmatrix}0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\\ *&*&*&*&*&*&*\\ *&0&0&0&0&0&0\\ *&0&0&0&*&*&*\\ *&0&0&0&*&0&*\end{pmatrix}.$$

In general, if $J=\operatorname{diag}(J_1,\dots,J_\nu)$, where $J_1,\dots,J_\nu$ denote Jordan blocks corresponding to different eigenvalues, then X_2^2 is a block-diagonal matrix $X_2^2=\operatorname{diag}((X_2^2)_1,\dots,(X_2^2)_\nu)$, each block $(X_2^2)_i$ corresponding to J_i as above.

(II.2.4)

EXAMPLE. For $n = 15$, $m = 5$, $r = 3$, $k_1 = 6$, $k_2 = 4$, $k_3 = 1$, $\delta_1(\lambda) = \delta_2(\lambda) = 2$, the above conditions on X, Y are

$$X = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\ *&0&0&0&0&0&*&0&0&0&*&0&0&0&0\\ *&0&0&0&0&0&*&0&0&0&*&*&*&*&*\\ *&0&0&0&0&0&*&0&0&0&*&*&0&0&0\\ *&0&0&0&0&0&*&0&0&0&*&*&0&*&* \end{pmatrix}$$

$$Y = \begin{pmatrix} 0&0&0&*&*\\ 0&0&*&*&*\\ 0&0&*&*&*\\ 0&0&*&*&*\\ 0&*&*&*&*\\ 0&0&0&0&0\\ 0&0&0&*&*\\ 0&0&*&*&*\\ 0&0&*&*&*\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&*&*\\ 0&0&0&*&*\\ 0&0&0&*&*\\ 0&0&0&*&* \end{pmatrix}$$

where $*$ denotes independent parameters.

III. SOME APPLICATIONS TO LOCAL PERTURBATIONS

III.1

The matrix $(A, B) + (X, Y)$ has not, in general, the same Brunovsky-Kronecker form as that of (A, B). In order to study the influence of the parameters of (X, Y) in the reduced form of $(A, B) + (X, Y)$, we are going to consider separately the incidence of the parameters according to whether they belong to X_1^2, Y_1^1, Y_2^1, or Y_2^2, respectively.

(i) If $y_2^1 \neq 0$ or $y_2^2 \neq 0$, it is clear that

$$\operatorname{rank}(B + Y) > r,$$

so that *the number of nilpotent blocks increases.*

(ii) Assume $Y_2^1 = 0$, $Y_2^2 = 0$, $Y_1^1 \neq 0$. This means that there exists k_i such that $k_i \geqslant k_{i+1} + 2$. We can suppose that

$$(A, B) + (X, Y) = \left(\begin{pmatrix} N_i & 0 \\ 0 & N_j \end{pmatrix}, \begin{pmatrix} E_i & Z \\ 0 & E_j \end{pmatrix} \right),$$

where $j = i + 1$ and $Z = (0, \dots, 0, z_1, \dots, z_{k_i - k_j - 1}, 0)^t$. Then, if the indices of controllability of this pair are k'_i, k'_j, it is easy to see that

$$k'_i - k'_j < k_i - k_j.$$

That is to say, if $Y_1^1 \neq 0$, *the differences $k_i - k_{i+1}$ tend to decrease*. In other words, *the indices of controllability tend to differ at most by* 1.

(iii) Assume $Y = 0$, $X_1^2 \neq 0$. We suppose that

$$(A, B) + (X, Y) = \left(\begin{pmatrix} N_i & 0 \\ X_1^2 & J \end{pmatrix}, \begin{pmatrix} E_i \\ 0 \end{pmatrix} \right).$$

Then the index of controllability of this new pair is $k_i + 1$. So, if $X_1^2 \neq 0$, *the size of the Jordan block decreases.*

III.2

From the previous discussion it follows that:

PROPOSITION. *With the above notation, if $(X, Y) \neq 0$, then the pair $(A,B) + (X, Y)$ is not block-similar to (A, B).*

III.3

We recall that a pair (A, B) is called *structurally stable* if there is a neighborhood $\mathscr{U}$ of (A, B) such that any pair in $\mathscr{U}$ is block-similar to (A, B).

From the above proposition we obtain the following characterization due to J. C. Willems [11, Theorem (6.5)]:

PROPOSITION. *Let c and d be integers such that $n = mc + d$. Then (A, B) is structurally stable if and only if $r = \min\{n, m\}$, $k_1 = \cdots = k_d = c + 1$, $k_{d+1} = \cdots = k_m = c$.*

Proof. We know that any pair in a sufficiently small neighborhood $\mathscr{U}$ of (A, B) is block-similar to some pair arising in a versal deformation of (A, B), that is, to some pair of the form $(A, B) + (X, Y)$ in (II.2.2).

Therefore, from the above proposition, a pair (A, B) is structurally stable if and only if the minimal deformation $(A, B) + (X, Y)$ of (A, B) in (II.2.2) is such that $(X, Y) = 0$—or, equivalently, the dimension in (II.1.9) is zero. In particular, $\delta_i(\lambda) = 0$ for all i, λ. That is equivalent to $n = p$. In addition, we must have $r = \min\{n, m\}$ and $k_1 \leqslant k_r + 1$. Conversely, the conditions in the proposition imply that the dimension in (II.1.9) is zero. ■

III.4

As the above proposition shows, the knowledge of the versal deformation of a pair (A, B) gives us a method for investigating the possible Brunovsky-Kronecker form of a perturbation of (A, B). The investigation of such a form leads us to consider the partition of the space of parameters into subsets (strata) with the following property: for all the values of the parameters belonging to the same strata, the resulting pairs have the same indices of controllability and their Jordan blocks are of the same type (that is, the eigenvalues can differ, but the number of distinct eigenvalues and the list of sizes of Jordan blocks corresponding to different eigenvalues are the same).

As an example we consider the pair

$$(A, B) = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right), \qquad (**)$$

whose miniversal deformation in (II.2.2) is

$$(A, B) + (X, Y) = \left(\begin{pmatrix} 0 & 0 & 0 \\ x & \lambda & 1 \\ y & z & \lambda + t \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right).$$

The controllability matrix of this pair is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & x & \lambda x + y \\ 0 & y & zx + (\lambda + t) y \end{pmatrix},$$

so that $(A, B) + (X, Y)$ is completely controllable if and only if

$$\begin{vmatrix} x & \lambda x + y \\ y & zx + (\lambda + t) y \end{vmatrix} = zx^2 + xyt - y^2 \neq 0,$$

this is to say, for all $(x, y, z, t) \in \mathbf{C}^4$ not belonging to the variety

$$zx^2 + xyt - y^2 = 0.$$

If we make the change of coordinates defined by

$$x = u,$$

$$z = w - \frac{t^2}{4},$$

$$y = v + u\frac{t}{2},$$

the above equation become

$$u^2 w - v^2 = 0,$$

which is the well-known Whitney umbrella. So in the exterior of the cylinder in $\mathbf{C}^4$ obtained by moving this Whitney umbrella along the t-axis, the corresponding pair is completely controllable.

We leave the other cases to the reader.

We collect in the following proposition the resulting classification.

PROPOSITION. *If (A, B) is block-similar to the pair $(**)$ above, then there exists a neighborhood $\mathcal{U}$ of (A, B) such that any pair in $\mathcal{U}$ is* *block-similar to one of the pairs listed below*:

$$\text{(i)}\quad \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda' & 1 \\ 0 & 0 & \lambda' \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)$$

$$\text{(ii)}\quad \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & \lambda' & 0 \\ 0 & 0 & \lambda'' \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right),$$

$$\text{(iii)}\quad \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \lambda' \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right),$$

$$\text{(iv)}\quad \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right).$$

Moreover, these cases correspond respectively to the following values of the above parameters u, v, and w, the variable t being arbitrary:

(i′) $u = v = w = 0$.
(ii′) $u = v = 0$, $w \neq 0$.
(iii′) $u^2 w - v^2 = 0$; $u \neq 0$ *or* $v \neq 0$.
(iv′) $u^2 w - v^2 \neq 0$.

Notice that this corresponds to the standard stratification of the Whitney umbrella [2, p. 4, example iii].

We are grateful to the referees for their suggestions and their careful review.

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Received 6 July 1993; final manuscript accepted 1 October 1995